Numerical solutions for elastic wave equations in layered media with perfectly matched layers

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Abstract

We present the analysis and numerical simulations of the perfectly matched layer (PML) in a discontinuous elastic medium. We prove that all interface wave modes in piecewise constant elastic media, separated by a planar interface, are dissipated by the PML. In addition, we present numerical simulations to verify the analysis and generalise the results to complex elastic media. Numerical examples using the Marmousi model demonstrates the utility of the PML and our numerical method for seismological applications.

Keywords: Elastic Waves, Perfectly Matched Layers, Stability Analysis, Finite Difference Methods, Summation by Parts

1 Summary

Consider the PML [3], in Cartesian coordinates $(x, y) \in \Omega \subset \mathbb{R}^2$,

$$\begin{aligned}
&\rho\left(\frac{\partial^2 \vec{\mathbf{u}}}{\partial t^2} + \sigma\frac{\partial \vec{\mathbf{u}}}{\partial t} - \sigma\alpha(\vec{\mathbf{u}} - \vec{\mathbf{q}})\right) = \frac{\partial}{\partial x}\vec{\mathbf{T}}_x + \frac{\partial}{\partial y}\vec{\mathbf{T}}_y,\\
&\frac{\partial \vec{\mathbf{v}}}{\partial t} = -(\sigma+\alpha)\vec{\mathbf{v}} + \frac{\partial \vec{\mathbf{u}}}{\partial x},\\
&\frac{\partial \vec{\mathbf{w}}}{\partial t} = -(\sigma+\alpha)\vec{\mathbf{w}} + \frac{\partial \vec{\mathbf{u}}}{\partial y},\\
&\frac{\partial \vec{\mathbf{q}}}{\partial t} = -\alpha(\vec{\mathbf{u}} - \vec{\mathbf{q}}),
\end{aligned} \tag{1}$$

with the PML stress vectors

$$\vec{\mathbf{T}}_x = A\frac{\partial \vec{\mathbf{u}}}{\partial x} + C\frac{\partial \vec{\mathbf{u}}}{\partial y} - \sigma A\vec{\mathbf{v}}, \tag{2}$$

$$\vec{\mathbf{T}}_y = C^T\frac{\partial \vec{\mathbf{u}}}{\partial x} + B\frac{\partial \vec{\mathbf{u}}}{\partial y} + \sigma B\vec{\mathbf{w}}. \tag{3}$$

The PML (1)–(3) absorbs outgoing waves in the x-direction in an elastic medium. The unknowns are the displacement vector $\vec{\mathbf{u}} \in \mathbb{R}^2$ and the PML auxiliary variable $\vec{\mathbf{v}}, \vec{\mathbf{w}}, \vec{\mathbf{q}} \in \mathbb{R}^2$. Here $\sigma(x) \geq 0$ is the PML damping function and $\alpha > 0$ is the complex frequency shift (CFS). To simplify the presentation we consider isotropic elastic medium defined by the density $\rho > 0$ and the coefficient matrices

$$A = \begin{bmatrix} 2\mu+\lambda & 0 \\ 0 & \mu \end{bmatrix},\ B = \begin{bmatrix} \mu & 0 \\ 0 & 2\mu+\lambda \end{bmatrix},\ C = \begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}$$

with λ, μ denoting the Lamé parameters. The density of the material is denoted by $\rho > 0$. The p and s wave speeds are given by $c_p = \sqrt{2\mu + \lambda/\rho}$, $c_s = \sqrt{\mu/\rho}$. In general the elastic media is heterogeneous and discontinuous. At discontinuities we define the outward unit normal vector $(n_x, n_y)^T \in \mathbb{R}^2$ on the interface and the PML traction vector $\vec{\mathbf{T}} = n_x\vec{\mathbf{T}}_x + n_y\vec{\mathbf{T}}_y$ and we enforce the jump conditions,

$$[\![\vec{\mathbf{T}}]\!] = 0, \quad [\![\vec{\mathbf{u}}]\!] = 0. \tag{4}$$

Note that the if the discontinuity lies on the y-axis (x-axis) we have $(n_x, n_y) = (0, 1)$ ($(n_x, n_y) = (1, 0)$). The problem is specified along with the initial conditions

$$\vec{\mathbf{u}}(0) = \mathbf{u}_0(x, y), \quad \frac{\partial \vec{\mathbf{u}}}{\partial t}(0) = \mathbf{u}_1(x, y). \tag{5}$$

Theorem 1 *Consider the PML model* (1)–(3) *in two half-plane elastic media with piecewise constant material parameters and the interface condition* (4) *at the planar interface, $y = 0$. If $\sigma > 0$ and $\alpha > 0$ are constants, then all interface wave modes are dissipated by the PML.*

The proof can be found in [4], and the result can be extended to anisotropic elastic medium as long as the corresponding whole plane constant coefficient PML is stable.

2 Results

2.1 Example 1

The first example considers a two-layer isotropic media on a Cartesian grid separated by a flat interface. The material constants were computed using the wave speeds $c_{s_1} = 1.8$, $c_{p_1} = 3.118$ on Layer 1 and $c_{s_2} = 3$, $c_{p_2} = 5.196$ on Layer 2. The densities were assumed to be $\rho_1 = 1.5$ and $\rho_2 = 3$ on Layer 1 and 2, respectively. We use a smooth Gaussian initial pulse for the displacement and a zero initial condition for the velocity. We compare the PML solution with the elastic-wave solution using only the ABC without the PML. We use the 4th-

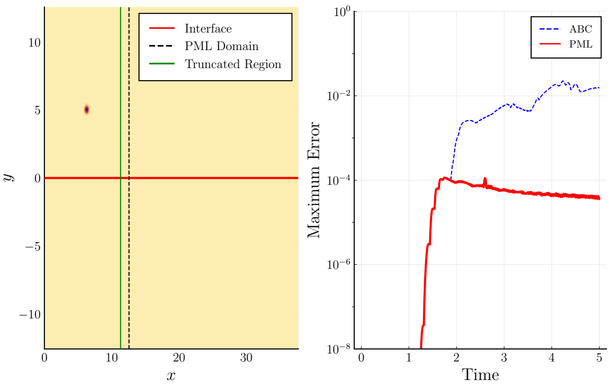

Figure 1: (Left) An example of a layered media along with the truncated region and the PML computational domain. Our goal is to restrict the computational domain using the PML (denoted by black, dashed line). The region to the left of the green line denotes the truncated domain, and the red dot inside the truncated domain is the smooth Gaussian initial pulse for the displacement. The reference solution is computed on the full domain (yellow region) using ABC. (Right) The maximum-norm error between the reference and PML solutions (red, solid) and the ABC solution (blue, dashed) inside the truncated region. The error is lower when PML is used, indicating that it is more effective than the absorbing boundary conditions.

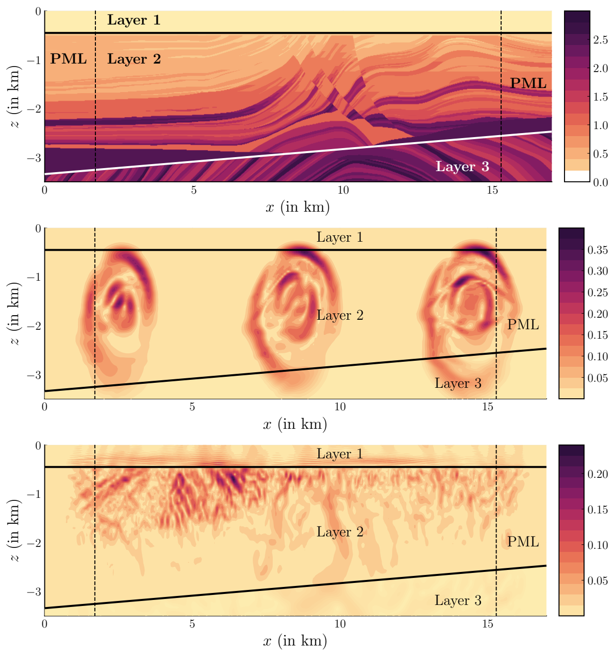

Figure 2: (Top) The shear wave speed c_s of the material obtained from the Marmousi2 dataset. (Middle) The solution at $T = 0.5$ to the PML model considered in Example 2 and (Bottom) The solution at $T = 10$. We observe that the wave speeds across the interfaces are different and the incoming waves are absorbed by the PML.

order summation-by-parts technique on a 481×161 grid to compute the reference solution on a larger domain. To discretise the temporal axis, we use the 4th-order Runge-Kutta scheme with $\Delta t = 0.2h/\sqrt{\max_i \left(c_{p_i}^2 + c_{s_i}^2\right)}$ and solve till final time $T = 10$. The extended domain is obtained by extending the PML domain three times along the $+x$ direction. We then compute the PML & ABC solution error inside the truncated region. The results are summarised in Figure 1.

2.2 Example 2

We solve the governing equations with the material properties obtained from the Marmousi2 dataset [1], which contains the material density and the wave speeds. Let L be the length of the domain. We define

$$\sigma(x) = \begin{cases} \sigma_0^v \left(\frac{L_{\sigma,1}-x}{\delta}\right)^3, & x < L_{\sigma,1} \\ \sigma_0^v \left(\frac{x-L_{\sigma,2}}{\delta}\right)^3, & x > L_{\sigma,2} \\ 0, & \text{o.w} \end{cases} \tag{6}$$

where $0 < L_{\sigma,1} < L_{\sigma,2} < L$. We compute the solution (shown in Figure 2) to (1)–(5) using the 4th-order summation-by-parts method in space on a 301×41 grid on Layers 1 and 3 and a 601×51 grid on Layer 2. We employ the 4th-order Runge-Kutta scheme in the temporal direction and solve till final time $T = 10\,\text{s}$. We consider a smooth Gaussian initial pulse for the displacement at three different locations in Layer 2, and a zero initial condition for the velocity and the auxiliary variables.

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