Applications of bi-orthogonal systems for the virtual element method

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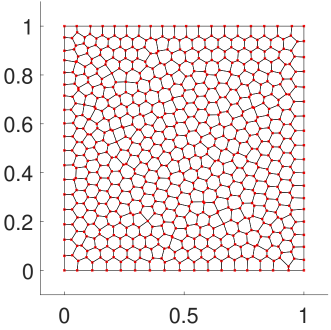
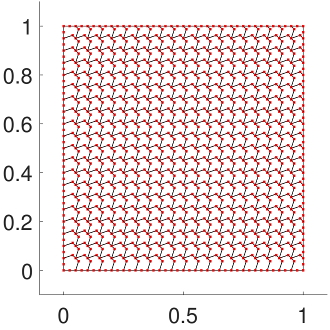
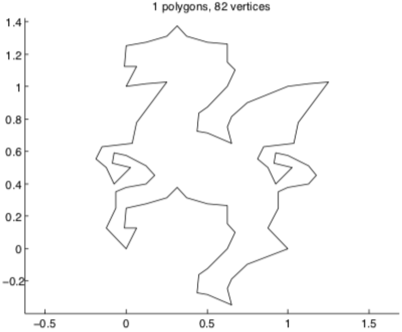
Outline

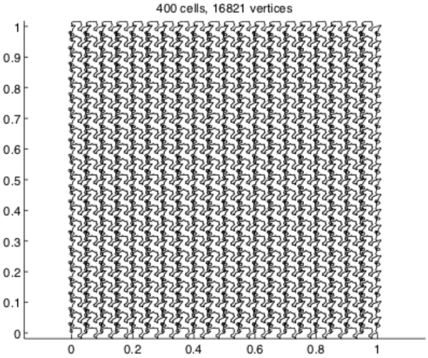
What is Virtual Element Method ?





A Pegasus?


1 polygons, 82 vertices
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.5
0
0.5
1
1.5


400 cells, 16821 vertices
1
0.9
0.8
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0.6
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0.8
1

Motivation

Computational Efficiency

- Virtual element method offers flexibility in mesh generation and adaptivity.
- Super–convergent gradient recovery.
- Efficient recovery of gradient, involves inverting a diagonal matrix.
- Mixed method for fourth order problem relaxes continuity requirement on the approximation.
- Efficient recovery of solution (involves inverting diagonal matrices) with good approximation properties.

Poisson Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let $\partial\Omega$ denote the boundary of the domain. We consider the Poisson problem

$$-\Delta u = f \quad \text{in} \quad \Omega,$$

subject to the boundary conditions

$$u = g \quad \text{on} \quad \partial\Omega.$$

Weak Formulation

The weak formulation of the problem is to find $u \in H^1(\Omega)$ such that

$$a(u, v) := (\nabla u, \nabla v) = (f, v)$$

for all $v \in H_0^1(\Omega)$, where $(\cdot, \cdot)$ denotes the L^2–innerproduct on Ω.

Projection operators I

Let $\mathcal{T}_h$ be a non-overlapping decomposition of polygons K and

$$\tilde{V}_K = \left\{ v \in H^1(K) \cap C^0(\partial K) \mid \Delta v \in \mathbb{P}_1(K),\ \ v|_e \in \mathbb{P}_1(e)\ \forall e \in \partial K \right\}.$$

The Ritz–projection

We define the Ritz–projection operator $\Pi_K^\nabla : \tilde{V}_K \to \mathbb{P}_1(K)$

$$\left(\nabla \Pi_K^\nabla v, \nabla p\right)_K = \left(\nabla v, \nabla p\right)_K \quad \text{and} \quad P_0(\Pi_K^\nabla v) = P_0(v)$$

for all $p \in \mathbb{P}_1(K)$ and

$$P_0(v) = \frac{1}{n_v} \sum_{i=1}^{n_v} v(V_i).$$

Projection operators II

The enriched local virtual element space W_K is given by

$$W_K = \left\{ v \in \tilde{V}_K \mid \left(\Pi_K^{\nabla} v, p\right)_K = (v, p)_K, \ \forall p \in P_1(K) \right\},$$

The L^2–projection

Define the L^2–projection $\Pi_K^0 : W_K \to \mathbb{P}_1(K)$ as

$$\left(\Pi_K^0 v, p\right)_K = (v, p)_K, \qquad \forall p \in \mathbb{P}_1(K).$$

The global space V_h is given by the assembly of the local spaces,

$$V_h = \left\{ v \in H^1(\Omega) \;:\; v|_K \in W_K \;\; \forall K \in \mathcal{T}_h \right\}.$$

Discrete Weak Formulation

The discrete weak formulation of the problem is to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = f(v_h)$$

for all $v_h \in V_h$.

Discrete bilinear forms

We have

$$a_h(u,v) = \sum_{K \in \mathcal{T}_h} a_h^K(u,v), \quad m_h(u,v) = \sum_{K \in \mathcal{T}_h} m_h^K(u,v)$$

where

$$a_h^K(u,v) = \left(\nabla \Pi_K^{\nabla} u, \nabla \Pi_K^{\nabla} v\right)_K + \sum_{i=1}^{n_v} \mathsf{dof}_i(u - \Pi_K^{\nabla} u)\, \mathsf{dof}_i(v - \Pi_K^{\nabla} v)$$

$$m_h^K(u,v) = \left(\Pi_K^{0} u, \Pi_K^{0} v\right)_K + |K| \sum_{i=1}^{n_v} \mathsf{dof}_i(u - \Pi_K^{0} u)\, \mathsf{dof}_i(v - \Pi_K^{0} v)$$

Equivalent L^2 and H^1 norms

It can be shown that the bilinear forms are bounded and coercive (Vacca, G. and Beirão da Veiga, L., 2015). For any function $w \in V_h$, define

$$\|w\|_{0,h} = \sqrt{m_h(w,w)},$$
$$|w|_{1,h} = \sqrt{a_h(w,w)}.$$

The discrete norms are equivalent to the L^2 and H^1 norms.

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Gradient recovery

Gradient Recovery

Bi-orthogonal bases:

Let $M_h = \text{span}\{\mu_1, \mu_2, \cdots, \mu_N\}$ be a space where μ_j satisfies the bi–orthogonal relation

$$\left(\Pi_K^0 \varphi_i,\, \Pi_K^0 \mu_j\right)_K = c_j \delta_{ij}$$

Gradient Recovery:

For $(x_1, x_2) = (x, y)$, the gradient recovery operator projects ∇u_h by finding $g_h^k = \mathsf{Q}_h(\partial u_h / \partial x_k) \in V_h$ such that

$$\sum_K \left(\Pi_K^0 g_k^h,\, \Pi_K^0 \mu_j\right)_K = \sum_K \left(\frac{\partial(\Pi_K^0 u_h)}{\partial x_k}, \Pi_K^0 \mu_j\right)_K$$

Kalyanaraman et al. (2019)

A gradient recovery method based on an oblique projection for virtual element method. *ANZIAM Journal, 60, 187-200.*

Theorem

(Stability) For $v \in L^2(\Omega)$, *we have*

$$\|Q_h v\|_{0,h} \leq \|v\|_{0,h},$$

and for $w \in H^1(\Omega)$, *we have*

$$|Q_h v|_{1,h} \leq |v|_{1,h}.$$

We have

$$\mathsf{D}_{ij}^{K} = \int_K \Pi_K^0 \varphi_i \, \Pi_K^0 \mu_j = \int_K \left(\sum_{\alpha=1}^{n_k} s_\alpha^i m_\alpha \right) \Pi_K^0 \mu_j ,$$

$$= \sum_{\alpha=1}^{n_k} s_\alpha^i \int_K m_\alpha \, \Pi_K^0 \mu_j ,$$

$$= \sum_{\alpha=1}^{n_k} s_\alpha^i \sum_{k=1}^{n_v} \mathsf{dof}_k(m_\alpha) \left(\int_K \varphi_k \, \Pi_K^0 \mu_j \right) .$$

In matrix form

$$\mathrm{D}^K = \left(\mathrm{D}^K \, \mathrm{M} \, \Pi_*^0 \right) .$$

Gradient recovery - Redefined

Bi–orthogonal bases:

Let $M_h = \text{span}\{\mu_1, \mu_2, \cdots, \mu_N\}$ be a space where μ_j satisfies the bi–orthogonal relation

$$\int_K \varphi_i \mu_j = c_j \delta_{ij} \quad \forall K \in \mathcal{T}_h.$$

Gradient recovery:

For $(x_1, x_2) = (x, y)$, the gradient recovery operator projects ∇u_h by finding $g_h^k = \mathsf{Q}_h(\partial u_h / \partial x_k) \in V_h$ such that

$$\sum_K (g_h^k, \mu_j)_K = \sum_K \left(\frac{\partial u_h}{\partial x_k}, \mu_j \right)_K$$

Left hand side

On any element K, we have

$$\sum_K \left(g_h^k, \mu_j\right)_K = \sum_K \sum_{i=1}^{n_v} \mathsf{dof}_i\left(g_h^k\right)\left(\varphi_i, \mu_j\right)_K .$$

Left hand side

Assembling the local system, we have the left hand side equal to

$$\boxed{\mathrm{D}\,\vec{g_k}}$$

where D is the assembled matrix from the bi–orthogonal relation.

Right hand side

The right hand side yields

$$\sum_{K}\left(\frac{\partial u_h}{\partial x_k},\mu_j\right)_K=\sum_{K}\sum_{i=1}^{n_v}\mathsf{dof}_i\left(u_h\right)\left(\frac{\partial \varphi_i}{\partial x_k},\mu_j\right)_K.$$

The right hand side

We split the left hand side by writing $\varphi_i=\Pi^0_K\varphi_i+(\mathsf{I}-\Pi^0_K)\varphi_i$ as

$$\left(\frac{\partial \varphi_i}{\partial x_k},\mu_j\right)_K=\left(\frac{\partial\,\Pi^0_K\varphi_i}{\partial x_k},\mu_j\right)_K+\left(\frac{\partial\left(\mathsf{I}-\Pi^0_K\right)\varphi_i}{\partial x_k},\mu_j\right)_K$$

Consistency term

$$\left(\frac{\partial \varphi_i}{\partial x_k}, \mu_j\right)_K \approx \underbrace{\left(\frac{\partial\, \Pi^0_K \varphi_i}{\partial x_k}, \mu_j\right)_K}_{\text{Consistency term}} + \underbrace{\frac{|K|}{h_K}\left(\sum_{m=1}^{n_v} \mathsf{dof}_m\left(\left(\mathsf{I} - \Pi^0_K\right)\varphi_i\right)\, \mathsf{dof}_m(\mu_j)\right)}_{\text{Stability Term}}$$

Consistency term

Consistency term is computed exactly. In matrix form,

$$\left(\frac{\partial\, \Pi^0_K \varphi_i}{\partial x_k}, \mu_j\right)_K := \left[\mathsf{D}^K\, \mathsf{M}_k\, \Pi^0_*\right] = \mathsf{Q}^K_k$$

where $[\mathsf{M}_k]_{\alpha m} = \mathsf{dof}_m\left(\frac{\partial m_\alpha}{\partial x_k}\right)$.

Stability term

Stability term is found from the bi–orthogonality relation and **is approximated**.

$$c_j\delta_{ij} = \int_K \varphi_i\mu_j = \underbrace{\int_K (\Pi_K^0\varphi_i)\,\mu_j}_{:=\text{Computed Exactly}} + \underbrace{\int_K \left((\mathsf{I} - \Pi_K^0)\varphi_i\right)\,\mu_j}_{:=\text{Approximated}}.$$

In matrix form, we have

$$\mathsf{D}^K = \mathsf{Q}_0^K + |K|\,\mathsf{S}^K.$$

Stability term

Thus we have

$$\mathsf{S}^K = \frac{1}{|K|}\left(\mathsf{D}^K - \mathsf{Q}_0^K\right).$$

Thus, we have in the right hand side of the **gradient recovery equation**

$$\left(\frac{\partial \varphi_i}{\partial x_k}, \mu_j\right)_K = \mathsf{Q}_k^K + \frac{1}{h_K}\left(\mathsf{D}^K - \mathsf{Q}_0^K\right), \\ = \mathsf{G}_k^K,$$

which can be computed without the explicit knowledge of the bi–orthogonal bases.

Right hand side

Thus assembling the elemental matrices over all the elements, we have the right hand side

$$\boxed{\mathsf{G}_k\, \vec{u_h}}$$

where the vector $\vec{u_h}$ is the global virtual element solution.

The gradient recovery

Thus, the gradient recovery can be performed by solving the linear system

$$\mathsf{D}\, \vec{g^k} = \mathsf{G}_k\, \vec{u_h},$$

or

$$\vec{g^k} = \left(\mathsf{D}^{-1} G_k\right)\, \vec{u_h}.$$

for $k = 1, 2$. The matrix D is diagonal and can be inverted easily.

Mixed virtual element method

Mixed virtual element method

We consider the bi–harmonic equation subject to clamped boundary condition,

$$\Delta^2 u = f \quad \text{in} \;\; \Omega$$
$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \;\; \partial\Omega$$

We split the fourth order problem as follows:

$$\Delta u = \phi,$$
$$\phi = p,$$
$$\Delta p = f$$

and

$$u = \frac{\partial u}{\partial n} = 0$$

Weak formulation

Following Lamichhane (2011), we find $((u,\phi),p) \in \mathbf{V} \times H^1(\Omega)$ such that

$$
\begin{aligned}
a\left((u,\phi),(v,\psi)\right) + b\left((v,\psi),p\right) &= (f,v) \\
b\left((u,\phi),q\right) &= 0
\end{aligned}
$$

for all $((v,\psi),q) \in \mathbf{V} \times H^1(\Omega)$, where $\mathbf{V} = H_0^1(\Omega) \times L^2(\Omega)$ and

$$
\begin{aligned}
a((u,\phi),(v,\psi)) &= (\phi,\psi), \\
b((v,\psi),q) &= (\nabla v, \nabla q) + (\psi, q).
\end{aligned}
$$

Discrete weak formulation

Find $((u_h, \phi_h), p_h) \in \mathbf{S_h} \times V_h$ such that

$$\begin{aligned} m_h(\phi_h, \psi_h) + b_h((v_h, \psi_h), p_h) &= f_h(v_h), \\ b_h((u_h, \phi_h), q_h) &= 0. \end{aligned}$$

for all $((v_h, \psi_h), q_h) \in \mathbf{S_h} \times V_h$ where $\mathbf{S}_h = V_h \times M_h$ and

$$b_h\left((v_h, \psi_h), p_h\right) = a_h(v_h, p_h) + m_h(\psi_h, p_h).$$

Definition of M_h

We define the space $M_h =$ span $\{\mu_1, \mu_2, \cdots, \mu_N\}$ such that

$$(\varphi_i, \mu_j)_K = c_j \delta_{ij} \quad \forall K \in \mathcal{T}_h,$$

where $V_h =$ span $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$.

Linear system

Associate the matrices

$$\mathbf{K} \to a_h(v_h, p_h), \quad \mathbf{D} \to m_h(\psi_h, p_h), \quad \mathbf{M} \to m_h(\phi_h, \psi_h),$$

respectively. The solution can be obtained by solving

$$\begin{aligned} \left(\mathbf{K}^T (\mathbf{D}^{-1})^T \mathbf{M}\, \mathbf{D}^{-1} \mathbf{K}\right) u_h &= \mathbf{f} \\ \phi_h &= -\left(\mathbf{D}^{-1} \mathbf{K}\right) u_h \\ p_h &= -\left((\mathbf{D}^{-1})^T \mathbf{M}\right) \phi_h \end{aligned}$$

Coercivity of the bilinear form

Define the Kernel space of the bilinear form b_h as

$$\text{Ker}\,\mathcal{B}_h = \{(v_h, \psi_h) \in V_h \times M_h \ : \ b_h((v_h, \psi_h), q_h) = 0 \quad \text{for all } q_h \in V_h\}$$

Theorem

There exists a positive constant C such that

$$m_h(\phi_h, \phi_h) \geq C \left(|u_h|^2_{1,h} + \|\phi_h\|^2_{0,h} \right)$$

for all $(u_h, \phi_h) \in \text{Ker}\,\mathcal{B}_h$.

Coercivity of the bilinear form

Define the Kernel space of the bilinear form b_h as

$$\text{Ker}\,\mathcal{B}_h = \{(v_h, \psi_h) \in V_h \times M_h \ : \ b_h((v_h, \psi_h), q_h) = 0 \quad \text{for all } q_h \in V_h\}$$

Theorem

There exists a positive constant C such that

$$m_h(\phi_h, \phi_h) \geq C \left(|u_h|_{1,h}^2 + \|\phi_h\|_{0,h}^2 \right)$$

for all $(u_h, \phi_h) \in \text{Ker}\,\mathcal{B}_h$.

This requires the coercivity of the bilinear form m_h which is guaranteed by the addition of stability term.

Numerical Results

Gradient recovery

Model Problem

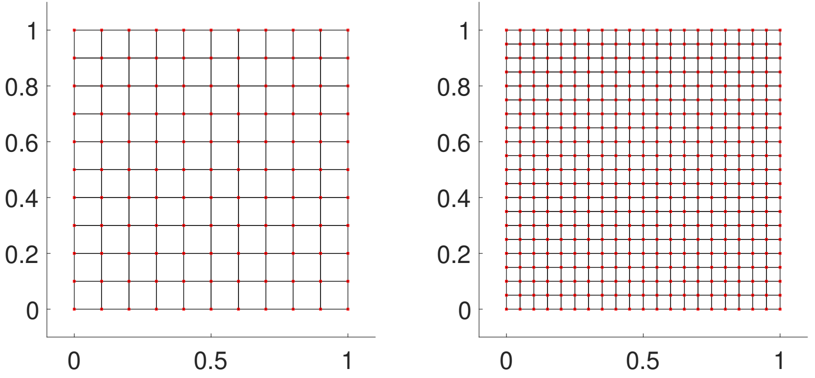
Consider the model problem

$$-\Delta u = f \quad \text{in} \quad \Omega = [0,1]^2$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

with $f = 2\pi^2 \sin(\pi x)\sin(\pi y)$. The exact solution is given by $u(x,y) = \sin(\pi x)\sin(\pi y)$.

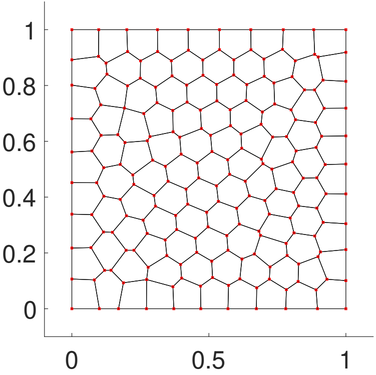
Linear virtual element method

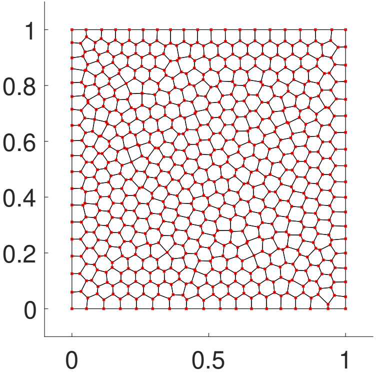
Square meshes.



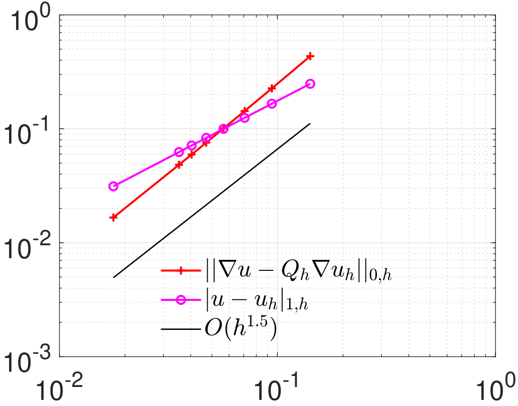
Linear virtual element method

Voronoi meshes.

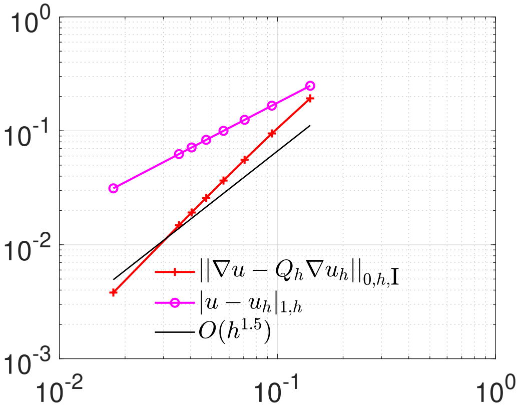




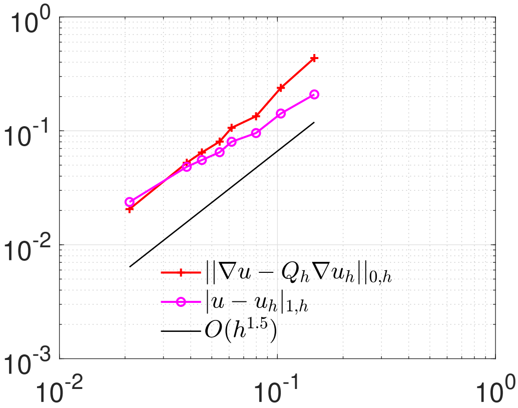
Square mesh



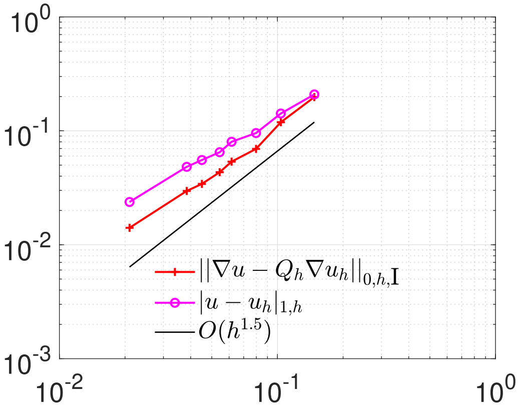
Square mesh - Interior



Voronoi mesh



Voronoi mesh - Interior



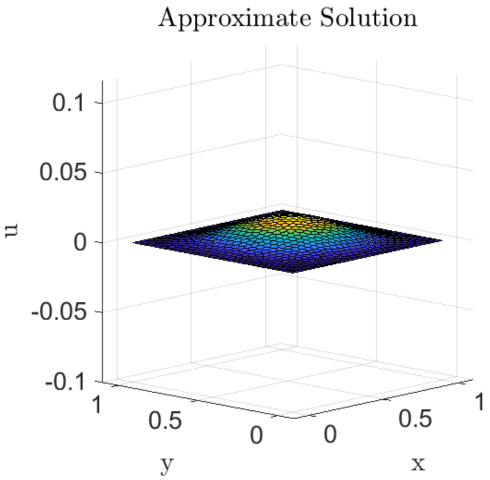
Mixed virtual element method

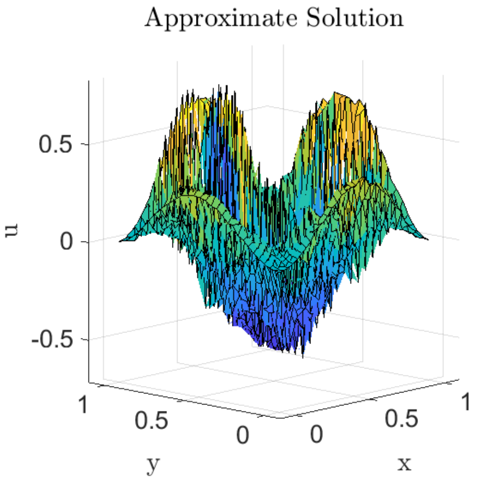
Model problem

Consider the model problem

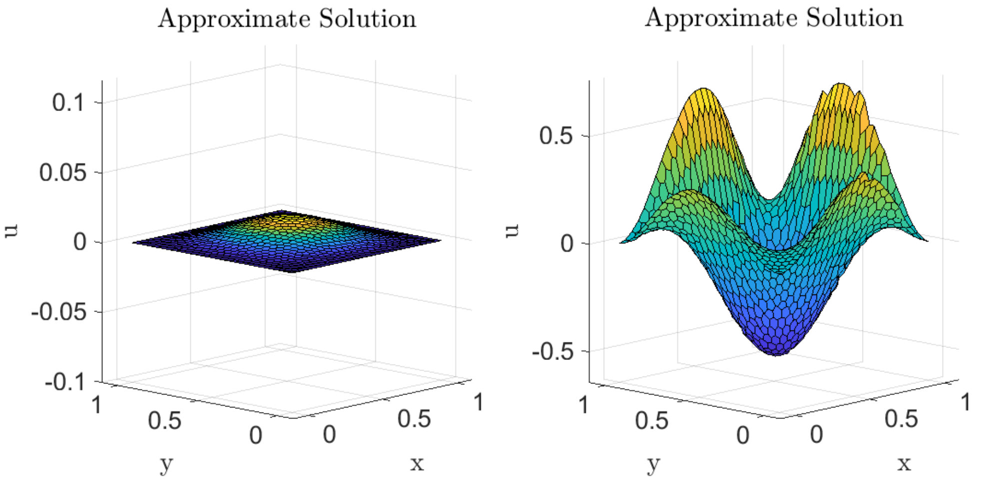
$$\Delta^2 u = f \quad \text{in} \quad \Omega = [0,1]^2$$
$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega.$$

The exact solution is given by $u(x,y) = (e^x + (x+1)e^y)\ x^2 y^2 (1-x)^2 (1-y)^2$.



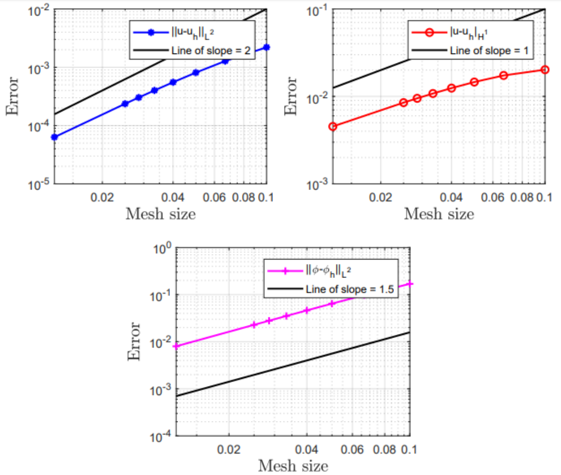


Solution (u, ϕ) without stability term.

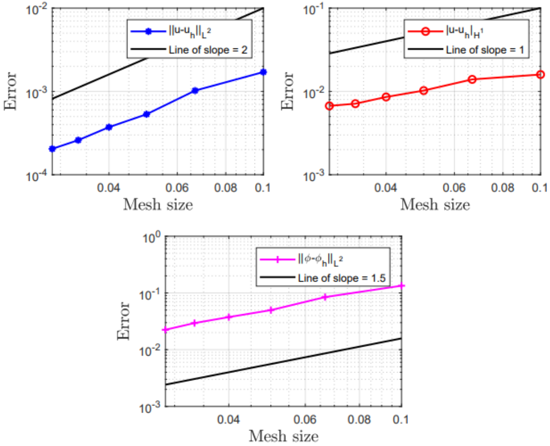


Solution (u, ϕ) with stability term.

Rates of convergence - Square



Rates of convergence - Voronoi



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