Vibrations of Ice Shelves

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Mathematics of Sea Ice and Ice Sheets, November 2020

https://balaje.github.io

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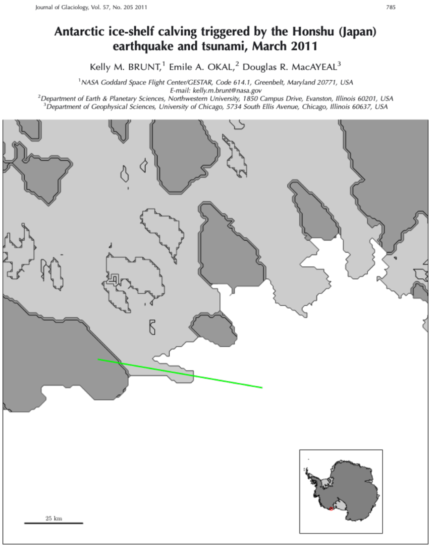
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Antarctic ice-shelf calving triggered by the Honshu (Japan) earthquake and tsunami, March 2011

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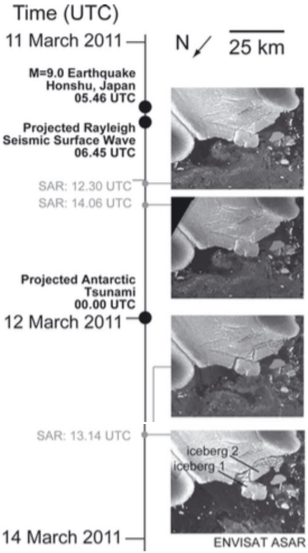

History

- First proposed by Holdsworth and Glynn (1978) in Nature.
- Holdsworth and Glynn (1981) studied the mechanism with data from the Erebus Glacier Tongue.
- Experimental measurements using seismometers were performed by MacAyeal et al. (2006); Cathles et al. (2009); Bromirski et al. (2010); Massom et al. (2018) which confirmed the presence of these vibrations.
- Mathematical models have been proposed to study the vibrations of ice–shelves, predominantly the thin–plate/shallow water models (Sergienko, 2013; Meylan et al., 2017) and more recently, numerical methods based on finite element methods have also been used (Papathanasiou et al., 2015a,b; Ilyas et al., 2018).

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Governing Equations

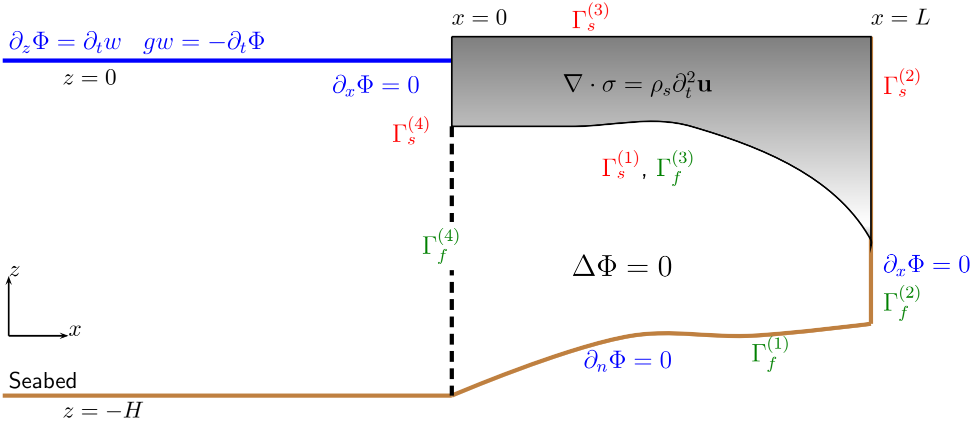

Figure: Governing equations as discussed by Kalyanaraman et al. (2020)

Finite element discretization

Define the following spaces

$$
\begin{aligned}
V &= H^1(\Omega_f),\\
W &= \left\{\mathbf{w} \;:\; \mathbf{w} \in \left[H^1(\Omega_s)\right]^2, \mathbf{w} = 0 \text{ on } \Gamma_s^{(2)}\right\},\\
V_h &= \left\{\phi_h \in V \;:\; \phi_h|_T \in \mathbb{P}_k(T) \text{ for all } T \in \mathcal{T}_f\right\},\\
W_h &= \left\{\mathbf{w}_h \in W \;:\; \mathbf{w}_h|_T \in [\mathbb{P}_k(T)]^2 \text{ for all } T \in \mathcal{T}_s\right\}
\end{aligned}
$$

The finite dimensional weak formulation of the coupled problem is to find $(\phi_h, \mathbf{w}_h) \in V_h \times W_h$ such that

$$
\begin{aligned}
(\nabla\phi_h, \nabla\psi)_{\Omega_f} &= -\mathrm{i}\omega \left\langle \mathbf{w}_h, \psi\right\rangle_{\Gamma_f^{(3)}} + \left\langle \mathbf{Q}\phi_h, \psi\right\rangle_{\Gamma_f^{(4)}} + \left\langle \chi, \psi\right\rangle_{\Gamma_f^{(4)}}\\
(\boldsymbol{\sigma}(\mathbf{w}_h) : \boldsymbol{\epsilon}(\mathbf{v}))_{\Omega_s} &= \rho_s\omega^2 (\mathbf{w}_h, \mathbf{v})_{\Omega_s} + \left\langle \mathbf{w}_h, \mathbf{v}\cdot\mathbf{n}\right\rangle_{\Gamma_s^{(1)}} -\mathrm{i}\omega \left\langle \phi_h, \mathbf{v}\right\rangle_{\Gamma_f^{(3)}}
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Modal expansion methods

To de–couple the ϕ_h and $\mathbf{w}_h$ in the problem, we assume the following ansatz for the final solution

$$\phi_h(x,z) = \phi_0(x,z) + \sum_{j=1}^{M} \lambda_j \phi_j(x,z), \quad \mathbf{w}_h(x,z) = \sum_{j=1}^{M} \lambda_j \boldsymbol{\eta}_j(x,z)$$

with λ_j's being the unknown "dofs". Substituting this into the weak formulation of the linear elasticity equations, we obtain

$$\sum_{j=1}^{M} \lambda_j \Big[(\boldsymbol{\sigma}(\boldsymbol{\eta}_j) : \boldsymbol{\epsilon}(\mathbf{w}_h))_{\Omega_s} - \rho_s \omega^2 (\boldsymbol{\eta}_j, \mathbf{w}_h)_{\Omega_s} - \langle \boldsymbol{\eta}_j, \mathbf{w}_h \cdot \mathbf{n} \rangle_{\Gamma_s^{(1)}} + \mathrm{i}\omega \langle \phi_j, \mathbf{w}_h \rangle_{\Gamma_f^{(3)}} \Big] = -\mathrm{i}\omega \langle \phi_0, \mathbf{w}_h \rangle_{\Gamma_f^{(3)}}$$

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Reduced System

This corresponds to the (reduced) linear system

$$\left[\mathbf{K} - \omega^2\mathbf{M} + \mathbf{C} + \mathrm{i}\omega\mathbf{B}\right]\boldsymbol{\lambda} = \mathbf{f}$$

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Properties

- Entries are analytic functions of ω.
- Can be interpolated as a function of ω once the finite element solutions are obtained on a coarse ω grid.
- Useful to construct time–domain solutions and complex resonances.

Diffraction and Radiation Potentials

The functions $\boldsymbol{\eta}_j \in W_h$ are the in–vacuo vibration modes of the ice–shelf which corresponds to solving the eigenvalue problem

$$(\boldsymbol{\sigma}(\boldsymbol{\eta}) : \boldsymbol{\epsilon}(\mathbf{v}))_{\Omega_s} = \rho_s \, \beta^2 \, (\boldsymbol{\eta}, \mathbf{v})_{\Omega_s}$$

for all $\mathbf{v} \in W_h$. The diffraction potential $\phi_0 \in V_h$ and the radiation potential $\phi_j \in V_h$ corresponding to the vibration mode $\boldsymbol{\eta}_j$ can be obtained by solving:

$$\begin{aligned}
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Properties

- Parallelizable.
- Useful to solve large problems involving multiple in–vacuo modes.

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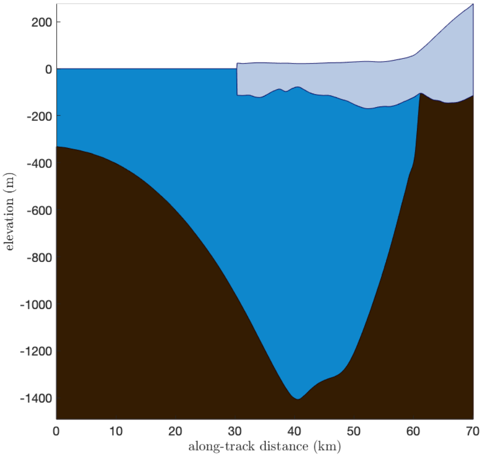
The Sulzberger Ice Shelf



The profiles were extracted from the BEDMAP2 dataset (Fretwell et al., 2013).

Sea Elevation Data

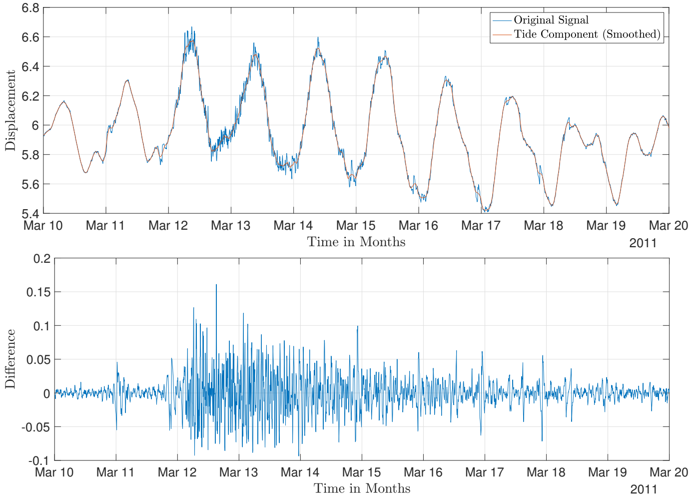
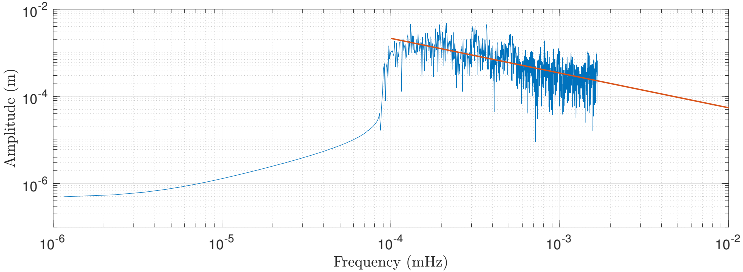


Figure: Tide gauge data from March 10 to March 20, 2009. Courtesy: Land Information New Zealand (LINZ)

Wave spectrum



Spectral analysis of the Tsunami wave indicates that the amplitude of the wave follows a power law (Rabinovich, 1997). The time–domain solution is obtained by the inverse Fourier transform,

$$\mathbf{u}(x,z,t) = \int_{-\infty}^{\infty} \hat{f}(x,\omega)\, e^{-\mathrm{i}(\omega t + \Phi_\omega)} \boldsymbol{\eta}(x,z,\omega)\, d\omega,$$

for a random phase $\Phi_\omega = \Phi_\omega(\omega)$ and $\hat{f}(x,\omega)$ is the power law spectrum shown in red.

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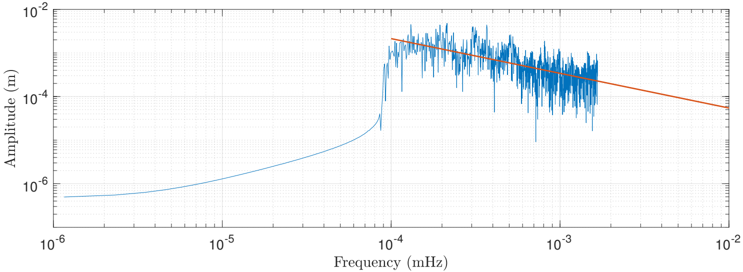

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Frequency–domain solutions

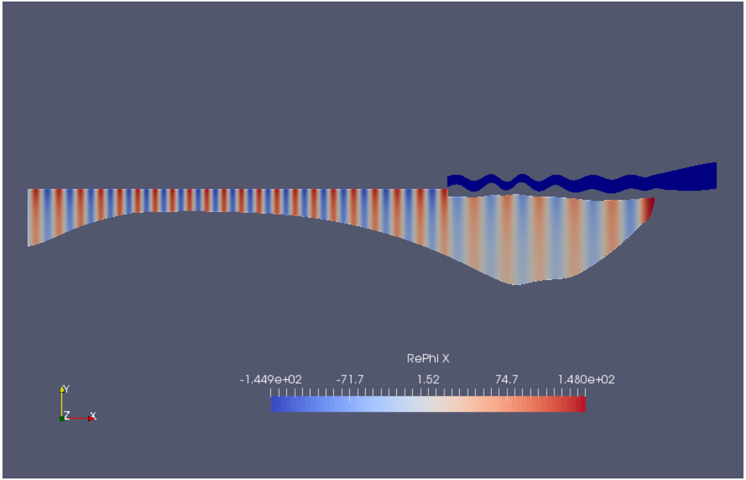


Figure: Frequency domain solution for $T = 50\,\mathrm{s}$. Value of reflection coefficient $R = -0.61272 - 0.79030\mathrm{i}$ and $|R| = 1$.

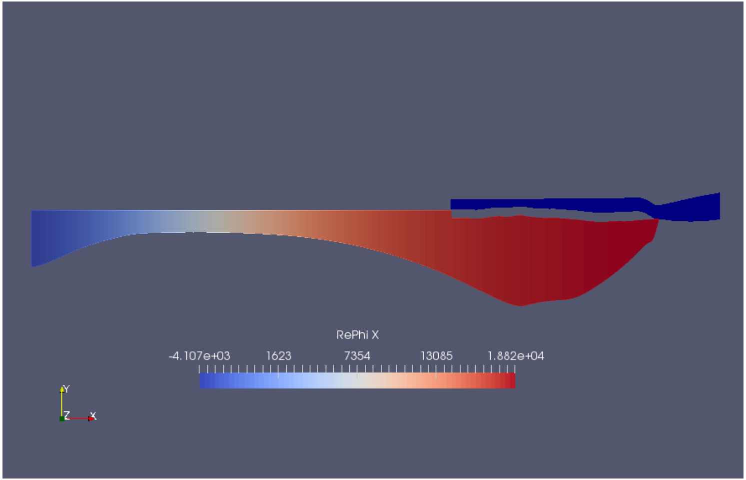
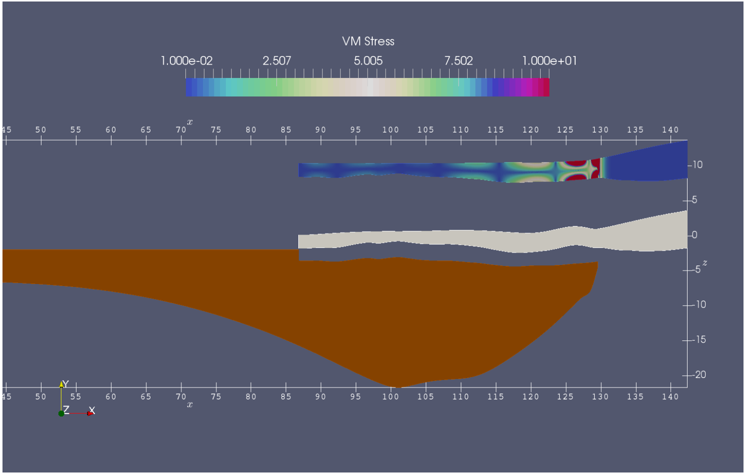


Figure: Frequency domain solution for $T = 5000\,\text{s}$. Value of reflection coefficient $R = -0.85007 - 0.52666\text{i}$ and $|R| = 1$.

Time–domain simulation



Conclusions

- A mathematical model based on linear elasticity and potential flow.
- Modal expansion methods to solve the resulting equations.
- Extension of these methods to solve real–life problems using BEDMAP2. All numerical experiments were performed using FreeFem (Hecht, 2012).

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A code repository written based on FreeFem is available on GitHub at https://github.com/Balaje/iceFem

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